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THE PROBLEM OF MINIMAX MEAN SQUARE FILTRATION IN PARABOLIC SYSTEMS

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The problem of estimating parameters of state of a distributed parabolic system by observation results is considered. The system is assumed to function under conditions of undefined perturbations in the measurement channel and specified initial distribution. The problem is considered in minimax formulation [1] in conformity with the scheme accepted for ordinary differential equations [2] (*). Analytic definition of sets $X(\vartheta, y(\cdot))$ $(\vartheta > 0)$ of states of a parabolic system compatible at instant ϑ with the realizable signal $y(t)$ $(t \in [0, \vartheta])$ is obtained. An element of region $X(\vartheta, y(\cdot))$ which satisfies the specified minimax criterion is chosen as the optimal estimate of the true state at instant ϑ. Integro-differential equations in partial derivatives are derived for parameters that define the evolution of regions $X(\vartheta, y(\cdot))$ in time. One of the methods of approximating the input problem of observation by similar problems for systems of ordinary differential equations is discussed on a specific example. Problems of observation for distributed systems in different formulations appear in [3 — 6].

1. Statement of the problem of a posteriori observation. Let some bounded region D with boundary S consisting of a finite number of $(n-1)$-dimensional hypersurfaces of class $C^3(D)$ ($C^p(D)$ is the set of all functions specified in D which have p continuous derivatives) be specified in the n-dimensional Euclidean space R^n. We consider in region D a system defined by the initial boundary value problem for the equation in partial derivatives of the parabolic type

$$\frac{\partial u(t, x)}{\partial t} = \Delta u(t, x) - q(x) u(t, x) \tag{1.1}$$

$$\alpha(\xi) u(t, \xi) + (1 - \alpha(\xi)) \frac{\partial u(t, \xi)}{\partial \nu} = 0$$

$$\lim_{t \to 0} \| u(t, x) - u_0(x) \|_{L_2(D)} = 0, \quad u_0(x) \in L_2(D)$$

$$\left(x = \operatorname{col}[x_1, \ldots, x_n] \in D,\ \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)$$

where t is the time, $t > 0$; $q(x)$ is a function continuous according to Hölder in

*) See A. B. Kurzhanskii and Iu. S. Osipov, Control and estimation problems in systems with distributed parameters. Preprints International Federation of Automatic Control, 6-th Triennial World Congress, Boston, 1975. Pittsburg, Pa., Instrument Society of America, 1975.

the compact region $D_1 = D \bigcup S$; $\xi \in S$, ν is the external normal to surface S at point ξ, and $\alpha(\xi)$ is a function of class $C^2(S)$ that satisfies condition $0 \leqslant \alpha(\xi) \leqslant 1$.

Let the signal $y(t)$ accessible to measurement on segment $[0, \vartheta]$ $(\vartheta > 0)$ be representable in the form

$$y(t) = \int_D \varkappa(x) u(t, x)\, dx + \eta(t), \quad \varkappa(x) \in L_2^m(D), \quad \eta(t) \in L_2^m(0, \vartheta) \quad (1.2)$$

where $\varkappa(x)$ is a known function; $\eta(t)$ is the error in the measurement channel, and $L_2^m(D)(L_2^m(0, \vartheta))$ is the transposed product of m spaces $L_2(D)$ $(L_2(0, \vartheta))$. Hence $y(t)$ is an m-dimensional function from $L_2^m(0, \vartheta)$.

Using the information provided by (1.2) we have to determine the true state of system (1.1) at instant ϑ, on the assumption that the initial state $u_0(x)$ and function $\eta(t)$ are not a priori known, but the condition

$$\beta^2 \int_D u_0(x) M(x) u_0(x)\, dx + \gamma^2 \int_0^\vartheta \eta'(t) N(t) \eta(t)\, dt \leqslant \mu^2 \quad (1.3)$$

which defines the region of their admissible values is specified. In this formula β, γ, and μ are some positive constants, $M(x)$ is a positive continuous in D_1 function, and $N(t)$ is a continuous $m \times m$ matrix positive definite for each $t \in [0, \vartheta]$, with the prime denoting transpostion.

D e f i n i t i o n 1. (See [2], Sect. 13). The set of those and only those states $u(\vartheta, x)$ of system (1.1) for each of which can be found functions $u_0(x)$ and $\eta(t)$ that satisfy relations (1.1) – (1.3) is called the information region $X(\vartheta, y(\cdot))$ of states compatible with the obtained signal $y(t)$ $(t \in [0, \vartheta])$.

D e f i n i t i o n 2. We call function $c(\vartheta, x)$ that satisfies the criterion

$$\varepsilon = \max_{z(\cdot)} \| z(\cdot) - c(\vartheta, \cdot \|_{L_2(D)}) = \min_{v(\cdot)} \max_{z(\cdot)} \| v(\cdot) - z(\cdot) \|_{L_2(D)} \quad (1.4)$$
$$z(\cdot),\ v(\cdot) \in X(\vartheta, y(\cdot))$$

the optimal estimate of the true state of system (1.1) at instant ϑ under conditions (1.2) and (1.3).

The determination of set $X(\vartheta, y(\cdot))$, and function $c(\vartheta, x)$ is the object of the problem of a posteriori observation [2].

Existence of the unique solution of problem (1.1) was shown in [6, 7] and to be of the form

$$u(t, x) = \int_D U(t, x, y) u_0(y)\, dy, \quad 0 < t < +\infty, \quad x \in D_1 \quad (1.5)$$

where $U(t, x, y)$ $(t > 0;\ x, y \in D_1)$ is the fundamental solution of system (1.1) that belongs to class C^1 with respect to t, to C^2 with respect to x, and to y from D_1.

We denote by $\{-\lambda_i, \omega_i(x), i = 1, 2, 3, \ldots\}$ the totality of eigenvalues and eigenfunctions of the elliptic operator in the right-hand side of Eq. (1.1) (with boundary condition in (1.1) satisfied). Then

a) $\lim\limits_{i\to\infty} \lambda_i = +\infty, \quad \lambda_i \geqslant \min\limits_{x\in D_1} q(x), \quad i = 1, 2, 3, \ldots,$

b) $\{\omega_i(x), i = 1, 2, 3, \ldots\}$ is the complete orthonormal system in $L_2(D)$

c) $$U(t, x, y) = \sum_{i=1}^{\infty} e^{-\lambda_i t}\omega_i(x)\omega_i(y)$$

where the series in the right-hand side uniformly converges on the arbitray set $[\delta, \infty) \times D_1 \times D_1$, $\delta > 0$;

d) $$\int_D U(t, x, y)\, h(y)\, dy = \sum_{i=1}^{\infty} e^{-\lambda_i t} h_i\omega_i(x)$$

$$\forall h(x) \in L_2(D), \quad h(x) = \sum_{i=1}^{\infty} h_i\omega_i(x)$$

and the series uniformly converges on the arbitrary set $[\delta, \infty) \times D_1$, $\delta > 0$.

Note that formula (1.5) and the last property imply the inclusion $u(t, x) \in L_2((0, T) \times D)$ for any $T > 0$.

2. Solution of the problem of a posteriori observation. We use the general procedure described in [2] and, first, define in space $L_2(D)$ the region $X(\vartheta, y(\cdot))$ in terms of appropriate support functionals, then, starting from condition (1.4), determine the sought function $c(\vartheta, x)$.

We rewrite formulas (1.2) and (1.5) in the form

$$u(\vartheta, \cdot) = Tu_0(x), \quad T: L_2(D) \to L_2(D)$$
$$y(\cdot) = T_0u_0(\cdot) + \eta(\cdot), \quad T_0: L_2(D) \to L_2^m(0, \vartheta)$$

It is clear from Sect. 1 that the operators T and T_0 are linear and continuous.

We introduce the notation $T_1 = T \times O$ and $T_2 = T_0 \times E$ (E and O are, respectively, the identical and the zero operators on $L_2^m(0, \vartheta)$), $z(\cdot) = \{u_0(\cdot), \eta(\cdot)\}$. The constraints (1.3) can now be represented in the form of inclusion $z(\cdot) \in Q$.

The definition of set $X(\vartheta, y(\cdot))$ implies that the element $u(\vartheta, \cdot) \in X(\vartheta, y(\cdot))$ then and only then when the following system of operator equations is compatible:

$$u(\vartheta, \cdot) = T_1z(\cdot), \quad y(\cdot) = T_2z(\cdot), \quad z(\cdot) \in Q$$

or, what is the same, when, by Theorem 3.1 in [2], the inequality

$$\min\{\langle T_1^*l(\cdot) + T_2^*\lambda(\cdot), z(\cdot)\rangle \mid z(\cdot) \in Q\} - \langle\lambda(\cdot), y(\cdot)\rangle \leqslant \langle l(\cdot), u(\vartheta, \cdot)\rangle \tag{2.1}$$

is satisfied for any $l(x) \in L_2(D)$, $\lambda(t) \in L_2^{m}\cdot(0, \vartheta)$. In this inequality $\langle(\cdot), (\cdot)\rangle$ denotes the scalar product in the respecitve Hilbert spaces and the asterisk denotes a conjugate operator.

Having determined the minimum (see [2], Sect. 13), from (2.1) for the supporting functional of set $X(\vartheta, y(\cdot))$ we obtain formula

$$\rho(l(\cdot)\,|\,X(\vartheta, y(\cdot))) = \inf_{\lambda(t)\in L_2^m(0,\vartheta)} \Big\{\int_0^\vartheta \lambda'(t)\,y(t)\,dt + \mu\Big[a^2(\vartheta) - 2\int_0^\vartheta f'(\vartheta, t)\,\lambda(t)\,dt + \int_0^\vartheta\int_0^\vartheta \lambda'(t)\,K(t,\tau)\,\lambda(\tau)\,dt\,d\tau + \frac{1}{\gamma^2}\int_0^\vartheta \lambda'(t)\,N^{-1}(t)\,\lambda(t)\,dt\Big]^{1/2}\Big\} \tag{2.2}$$

$$a^2(\vartheta) = \frac{1}{\beta^2}\iiint_{DDD} M^{-1}(y)\,U(\vartheta, x, y)\,l(x)\,U(\vartheta, \eta, y)\,l(\eta)\,d\eta\,dx\,dy$$

$$f(\vartheta, t) = \frac{1}{\beta^2}\iiint_{DDD} U(\vartheta, x, y)\,l(x)\,M^{-1}(y)\,\varkappa(\eta)\,U(t, \eta, y)\,d\eta\,dx\,dy$$

$$K(t, \tau) = \frac{1}{\beta^2}\iiint_{DDD} \varkappa(x)\,\varkappa'(\eta)\,U(t, x, y)\,M^{-1}(y)\,U(\tau, \eta, y)\,d\eta\,dx\,dy$$

Let us consider matrix $K(t, \tau)$ in more detail. Taking into account formulas (1.5) and (1.1), it is possible to show that this matrix is symmetric with respect to t and τ, semipositive definite (by construction), and continuous over the totality of variables in region $[0, \vartheta] \times [0, \vartheta]$ for arbitrary positive ϑ.

We set

$$\langle h_1(\cdot), h_2(\cdot)\rangle_K = \int_0^\vartheta\int_0^\vartheta h_1'(t)\,K(t,\tau)\,h_2(\tau)\,dt\,d\tau + \frac{1}{\gamma^2}\int_0^\vartheta h_1'(t)\,N^{-1}(t)\,h_2(t)\,dt$$

$$\forall h_1(t),\ h_2(t) \in L_2^m(0, \vartheta)$$

and consider the following system of Fredholm integral equations of the second kind with a nonnegative kernel (which has a unique solution in $L_2^m(0, \vartheta)$) [8, 9]:

$$\int_0^\vartheta K(t,\tau)\,d(\vartheta,\tau)\,d\tau + \frac{1}{\gamma^2}N^{-1}(t)\,d(\vartheta, t) = f(\vartheta, t) \tag{2.3}$$

$$\int_0^\vartheta K(t,\tau)\,y^*(\vartheta,\tau)\,d\tau + \frac{1}{\gamma^2}N^{-1}(t)\,y^*(\vartheta, t) = y(t)$$

The argument ϑ in functions $d(\vartheta, t)$ and $y^*(\vartheta, t)$ implies that Eqs. (2.3) are considered on segment $[0, \vartheta]$.

The right-hand side of formula (2.2) can now be expressed in the form

$$\rho(l(\cdot)\,|\,X(\vartheta, y(\cdot))) = \inf_{\lambda(t)\in L_2^m(0,\vartheta)} \{\langle\lambda(\cdot), y^*(\vartheta, \cdot)\rangle_K + \tag{2.4}$$

$$\mu\,[\langle\lambda(\cdot)-d(\vartheta,\cdot),\lambda(\cdot)-d(\vartheta,\cdot)\rangle_K+g^2(\vartheta)]^{1/2}\}$$
$$g^2(\vartheta)=a^2(\vartheta)-\langle d(\vartheta,\cdot),\,d(\vartheta,\cdot)\rangle_K$$

where $g^2(\vartheta)\geqslant 0$, since (by definition) the radicand in (2.4) is determinate for any function $\lambda(t)$ from $L_2^m(0,\vartheta)$.

Calculation of the lower bound in (2.4) (see [2], Section 13) yields

$$\rho\,(l(\cdot)\mid X(\vartheta,y(\cdot)))=g(\vartheta)(\mu^2-\langle y^*(\vartheta,\cdot),\,y^*(\vartheta,\cdot)\rangle_K^{1/2}+ \langle y^*(\vartheta,\cdot),\,d(\vartheta,\cdot)\rangle_K \tag{2.5}$$

Remark 1. Let us set

$$f(\vartheta,t,x)=\frac{1}{\beta^2}\int_D\int_D U(\vartheta,x,y)\,M^{-1}(y)\,\varkappa(\eta)\,U(t,\eta,y)\,d\eta\,dy$$

Then

$$f(\vartheta,t)=\int_D l(x)\,f(\vartheta,t,x)\,dx$$

The properties a), b) and c) defined in Sect. 1 imply that function $f(\vartheta,t,x)$ is continuous over the totality of variables in any arbitrary region $[\delta,\infty)\times[0,\vartheta]\times D_1$, $\delta>0$.

Let $d(\vartheta,t,x)$ be a solution which differs from that of the first of Eqs. (2.3) by that in the right-hand side it contains $f(\vartheta,t,x)$ instead of $f(\vartheta,t)$. This equation has also a unique solution for any $x\in D_1$. Note now that

$$d(\vartheta,t)=\int_D l(x)\,d(\vartheta,t,x)\,dx$$

(using the respective properties of solutions of the indicated type [8, 9] it can be shown that the above integral exists).

Using the notation

$$h^2(\vartheta)=\langle y^*(\vartheta,\cdot),\,y^*(\vartheta,\cdot)\rangle_K=\int_0^\vartheta y'(t)\,y^*(\vartheta,t)\,dt \tag{2.6}$$

$$P(\vartheta,x,y)=\frac{1}{\beta^2}\int_D U(\vartheta,x,\eta)\,M^{-1}(\eta)\,U(\vartheta,y,\eta)\,d\eta- \int_0^\vartheta d'(\vartheta,t,y)\,f(\vartheta,t,x)\,dt$$

$$c(\vartheta,x)=\int_0^\vartheta f'(\vartheta,t,x)\,y^*(\vartheta,t)\,dt=\int_0^\vartheta d'(\vartheta,t,x)\,y(t)\,dt= \langle y^*(\vartheta,\cdot),\,d(\vartheta,\cdot,x)\rangle_K$$

from formula (2.5) we obtain

$$\rho\,(l(\cdot)\,|\,X(\vartheta,y(\cdot)))=\left\{\int_D\int_D l(x)\,P(\vartheta,x,y)\,l(y)\,dx\,dy\right\}^{1/2}(\mu^2- \tag{2.7}$$

$$h^2(\vartheta))^{1/2} + \int_D l(x)\, c(\vartheta, x)\, dx$$

As shown in [10], the set with the supporting functional (2.7) is an ellipsoid (defined in space $L_2(D)$ by the related scalar product) whose center is at point $c(\vartheta, \cdot)$.

Thus the following statement is valid.

Theorem 1. The information region $X(\vartheta, y(\cdot))$ of states of system (1.1) that is compatible with the realized signal $y(t)$ $(t \in [0, \vartheta])$ with restraint (1.3) is an ellipsoid, possibly degenerate, with supporting functional (2.7) and center at point $c(\vartheta, x)$ defined by the last of formulas (2.6).

Let us consider the geometrical meaning of the derived solution of the a posteriori observation.

Let the obtained signal $y(t)$ generated by the pair of functions

$$u_0(x) = \sum_{k=1}^{\infty} a_k \omega_k(x), \quad \eta(t)$$

be such that (see (1.2))

$$y(t) = \varphi(t) + \eta(t), \quad \varphi(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} b_k a_k \quad \left(\varkappa(x) = \sum_{k=1}^{\infty} b_k \omega_k(x)\right) \tag{2.8}$$

We denote by L the set of all functions from $L_2^m(0, \vartheta)$ of the form $\varphi(t)$ such that it is possible to determine on the basis of function $u_0(x)$ some function $\eta(t)$ and a pair $\{u_0(x), \eta(t)\}$ which satisfy conditions (2.8) and (1.3). Let L_1 be the complement of L to $L_2^m(0, \vartheta)$. Formula (2.8) can then be expressed in the form

$$y(t) = y_1(t) + y_2(t), \quad y_1(t) = \varphi(t) + \eta_1(t) \in L$$
$$y_2(t) = \eta_2(t) \in L_1, \quad \eta(t) = \eta_1(t) + \eta_2(t)$$

The statement is proved (cf. [2], Sect. 11).

Theorem 2. 1°. The optimal estimate of the true state of system (1.1) at instant ϑ with conditions (1.2) and (1.3) is the center of ellipsoid $X(\vartheta, y(\cdot))$, function $c(\vartheta, x)$ defined by the last of formulas (2.6).

Let us consider the set of signals $y(t) = y_1(t) + y_2(t)$ $(y_1(t) \in L,\ y_2(t) \in L_1)$ generated in conformity with (1.1)–(1.3) with fixed first component $y_1(t)$.

2°. If the second component of the signal, i.e. function $y_2(t)$ from L_1 is zero, the information region $X(\vartheta, y(\cdot))$ reaches its maximum dimension (which is most unfavorable for the observer, since the estimate error is then maximal).

3°. In the most favorable case for the observer ellipsoid $X(\vartheta, y(\cdot))$ may degenerate into a point. Functions $u_0(x)$ and $\eta(t)$ that have generated $y(t)$ then satisfy constraint (1.3) with the equality sign.

3. Equations of maximum filtration. We pass to the investigation of dynamics of information regions $X(\vartheta, y(\cdot))$ using Theorems 1 and 2. We adduce the derivation of differential equations for functions $c(\vartheta, x)$, $P(\vartheta, x, y)$, and $h^2(\vartheta)$ the dynamics of whose variation determines the evolution of regions $X(\vartheta, y(\cdot))$. For this we carry a number of transformations whose admissibility will

be substantiated below.

We introduce the auxilliary function $B(\vartheta, x, y)$ defined by formula

$$B(\vartheta, x, y) = \langle d(\vartheta, \cdot, x), d(\vartheta, \cdot, y)\rangle_K \tag{3.1}$$

Then, differentiating function $B(\vartheta, x, y)$ with respect to ϑ and taking into account formulas (3.1) and (1.5), Remark 1, and also the equlities

$$\int_0^\vartheta K(t, \vartheta)\, d(\vartheta, t, x)\, dt = \int_0^\vartheta \int_D f(\vartheta, t, z)\, \varkappa'(z)\, d(\vartheta, t, x)\, dz\, dt = \int_D \varkappa(y)\, B(\vartheta, x, y)\, dy$$

we obtain for function $B(\vartheta, x, y)$ the problem

$$\begin{aligned} \frac{\partial B(\vartheta, x, y)}{\partial \vartheta} &= \Delta B(\vartheta, x, y) - [q(x) + q(y)]\, B(\vartheta, x, y) + \\ &\gamma^2 \Big[f(\vartheta, \vartheta, x) - \int_D \varkappa(y)\, B(\vartheta, x, y)\, dy \Big]' N(\vartheta) \Big[f(\vartheta, \vartheta, y) - \\ &\int_D \varkappa(x)\, B(\vartheta, x, y)\, dx \Big] \end{aligned} \tag{3.2}$$

$$\vartheta > 0; \quad x, y \in D$$

$$\alpha(\xi)\, B(\vartheta, x, \xi) + (1 - \alpha(\xi)) \frac{\partial B(\vartheta, x, \xi)}{\partial \nu} = 0, \quad \xi \in S$$

$$\alpha(\xi)\, B(\vartheta, \xi, y) + (1 - \alpha(\xi)) \frac{\partial B(\vartheta, \xi, y)}{\partial \nu} = 0, \quad \xi \in S$$

$$\lim_{\vartheta \to 0} \| B(\vartheta, x, y) \|_{L_2(D \times D)} = 0$$

From the second of formulas (2.6) follows that

$$P(\vartheta, x, y) = \frac{1}{\beta^2} \int_D U(\vartheta, x, \eta)\, M^{-1}(\eta)\, U(\vartheta, y, \eta)\, d\eta - B(\vartheta, x, y) \tag{3.3}$$

Hence function $P(\vartheta, x, y)$ satisfies the following initial boundary value problem:

$$\begin{aligned} \frac{\partial P(\vartheta, x, y)}{\partial \vartheta} &= \Delta P(\vartheta, x, y) - (q(x) + q(y))\, P(\vartheta, x, y) - \\ &\gamma^2 \int_D \int_D \varkappa'(\eta)\, P(\vartheta, x, \eta)\, N(\vartheta)\, P(\vartheta, y, z)\, \varkappa(z)\, d\eta\, dz, \quad \vartheta > \delta > 0 \end{aligned} \tag{3.4}$$

$$x, y \in D$$

$$\alpha(\xi)\, P(\vartheta, \xi, y) + (1 - \alpha(\xi)) \frac{\partial P(\vartheta, \xi, y)}{\partial \nu} = 0, \quad \xi \in S$$

$$\alpha(\xi)\, P(\vartheta, x, \xi) + (1 - \alpha(\xi)) \frac{\partial P(\vartheta, x, \xi)}{\partial \nu} = 0, \quad \xi \in S$$

$$P(\vartheta, x, y)\,|_{\vartheta = \delta} = P(\delta, x, y)$$

where δ is an arbitrary positive number and $P(\delta, x, y)$ is determined by formula (2.6).

Problems for functions $c(\vartheta, x)$ and $h^2(\vartheta)$ are similarly derived

$$\frac{\partial c(\vartheta, x)}{\partial\vartheta} = \Delta c(\vartheta, x) - q(x)\, c(\vartheta, x) + \gamma^2 \left[y(\vartheta) - \int_D c(\vartheta, x)\varkappa(x)\, dx\right]' \quad (3.5)$$

$$N(\vartheta)\left[f(\vartheta, \vartheta, x) - \int_D \varkappa(y) B(\vartheta, x, y)\, dy\right], \quad \vartheta > 0, \quad x \in D$$

$$\alpha(\xi)\, c(\vartheta, \xi) + (1 - \alpha(\xi)) \frac{\partial c(\vartheta, \xi)}{\partial\nu} = 0, \quad \xi \in S$$

$$\lim_{\vartheta\to 0} \| c(\vartheta, x) \|_{L_2(D)} = 0$$

$$\frac{\partial c(\vartheta, x)}{\partial\vartheta} = \Delta c(\vartheta, x) - q(x)\, c(\vartheta, x) + \gamma^2 \left(y(\vartheta) - \int_D c(\vartheta, x)\varkappa(x)\, dx\right)' \quad (3.6)$$

$$N(\vartheta) \int_D \varkappa(y) P(\vartheta, x, y)\, dy, \quad \vartheta > \delta > 0, \quad x \in D$$

$$\alpha(\xi)\, c(\vartheta, \xi) + (1 - \alpha(\xi)) \frac{\partial c(\vartheta, \xi)}{\partial\nu} = 0, \quad \xi \in S$$

$$c(\vartheta, x)\big|_{\vartheta=\delta} = c(\delta, x)$$

$$\frac{\partial h^2(\vartheta)}{\partial\vartheta} = \gamma^2 \left[y(\vartheta) - \int_D \varkappa(x)\, c(\vartheta, x)\, dx\right]' N(\vartheta) \left[y(\vartheta) - \right. \quad (3.7)$$

$$\left. \int_D \varkappa(x)\, c(\vartheta, x)\, dx\right]$$

Theorem 3. Functions $c(\vartheta, x)$, $P(\vartheta, x, y)$, and $h^2(\vartheta)$ are solutions of problems (3.6), (3.4) and (3.7).

Remark 2. Function $c(\vartheta, x)$ can be also determined as the solution of problem (3.5) in which $B(\vartheta, x, y)$ is the solution of problem (3.2). Functions $B(\vartheta, x, y)$ and $P(\vartheta, x, y)$ are linked by relationship (3.3).

The validity of above calculations can be substantiated by the corresponding solutions of Fredholm integral equations (2.3) of the second kind [8, 9]. It can also be shown that functions ((ϑ, x), $P(\vartheta, x, y)$, $B(\vartheta, x, y)$, and $h^2(\vartheta)$ are continuous over the totality of variables for $\vartheta > 0$, with $x, y \in D_1$, and have continuous derivatives $\partial^2 c(\vartheta, x)/\partial x_i^2$, $\partial^2 P(\vartheta, x, y)/\partial x_i^2$, $\partial^2 P(\vartheta, x, y)/\partial y_i^2$, $\partial P(\vartheta, x, y)/\partial\vartheta$, $\partial^2 B(\vartheta, x, y)/\partial x_i^2$, $\partial^2 B(\vartheta, x, y)/\partial y_i^2$, $\partial B(\vartheta, x, y)/\partial\vartheta$ ($\vartheta > 0$; $x, y \in D_1$, $i = 1, 2, \ldots, n$). Functions $c(\vartheta, x)$ and $h^2(\vartheta)$ are differentiable with respect to ϑ for almost all $\vartheta > 0$.

Using the investigations in [11] it is possible to show the unique solvability of the initial boundary value problems (3.6) and (3.4) in the considered class of functions.

4. The problem of approximation. Let us investigate on a specific example one of the possible method of approximating the a posteriori observation input problem (1.1) – (1.4), using similar problems for certain systems of ordinary

differential equations whose solutions are well known.

Let us consider the problem defined by the Dirichlet problem for the heat conduction equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2}, \quad x \in (0,1), \quad t > 0 \tag{4.1}$$
$$u(t,0) = u(t,1) = 0, \quad u(t,x)|_{t=0} = u_0(x) \in L_2(0,1)$$

We assume for definiteness that $M(x) = 1$ and $N(t)$ is a unit $m \times m$ matrix. In this case constraint (1.3) is of the form

$$\beta^2 \int_0^1 u_0^2(x)\,dx + \gamma^2 \int_0^{\vartheta} \eta'(t)\,\eta(t)\,dt \leqslant \mu^2 \tag{4.2}$$

As the approximating sequence for problems (4.1), (1.2), and (4.2) we consider the following set of a posteriori observation problems (see footnote on p. 1112). For the system

$$\frac{du_1^n}{dt} = \frac{1}{h^2}(-2u_1^n + u_2^n) \tag{4.3}$$
$$\frac{du_i^n}{dt} = \frac{1}{h^2}(u_{i-1}^n - 2u_i^n + u_{i+1}^n), \quad i = 2, \ldots, n-1$$
$$\frac{du_n^n}{dt} = \frac{1}{h^2}(u_{n-1}^n - 2u_n^n), \quad t > 0$$
$$u_i^n(0) = u_{0i}^n, \quad i = 1, \ldots, n \left(h = \frac{1}{n+1}\right)$$

and the equation of measurement

$$y(t) = G^n u^n(t) + \xi(t), \quad t \in [0, \vartheta] \tag{4.4}$$

we have to define in space R^n the information region $X_n(\vartheta, y(\cdot))$ of states of system (4.3) compatible with signal $y(t)$ $(t \in [0, \vartheta])$, i.e. the set of those and only those vectors $u^n(\vartheta) \in' R^n$ for each of which can be found a pair u_0^n, $\xi(t)$ that satisfies formulas (4.3) and (4.4) with condition

$$\beta^2 u_0^{n\prime} H^n u_0^n + \gamma^2 \int_0^{\vartheta} \xi'(t)\,\xi(t)\,dt \leqslant \mu^2 + \varepsilon_1, \quad \varepsilon_1 > 0 \tag{4.5}$$

The term G^n in (4.4) represents an $n \times m$ matrix with elements

$$g_{ji}^n = \int_{hi}^{h(i+1)} \varkappa_j(x)\,dx, \quad i = 0, \ldots, n-1, \quad j = 1, \ldots, m$$

$$u_0^n = \mathrm{col}\,[u_{01}^n, \ldots, u_{0n}^n], \quad H^n = \mathrm{diag}\,\{h, \ldots, h\}$$
$$\varkappa(x) = \mathrm{col}\,[\varkappa_1(x), \ldots, \varkappa_m(x)]$$

Note that function $y(t)$ appearing in (4.4) is one and the same for all n and represents precisely the signal that was realised at the output of system (4.1) by virtue of (1.2).

The admissibility of such formulation of problems(4.3)—(4.5) is validated by the addition to the right-hand side of inequality (4.5) of the positive arbitrary selected number ε_1.

By solving problems (4.3) —(4.5) (see [2], Sect. 13) we obtain the sequence of functions $h_n^2(\vartheta)$, $P^n(\vartheta)$ and $c^n(\vartheta)$ which serve as parameters of corresponding information regions $X_n(\vartheta, y(\cdot))$.

We introduce the notation

$$c^n(\vartheta, x) = \begin{cases} c_1^n(\vartheta), & 0 \leqslant x \leqslant h \\ c_i^n(\vartheta), & (i-1)h < x \leqslant ih \\ 0, & nh < x \leqslant 1 \end{cases} \tag{4.6}$$

and denote by $P^n(\vartheta, x, y)$ the function derived with the use of the $n \times n$ matrix $P^n(\vartheta)$ in a way similar to (4.6). The following theorem on approximation is valid.

Theorem 4. Let $\varkappa_j(x)$ $(j = 1, \ldots, m)$ be functions bounded on segment $[0, 1]$ $(\varkappa(x) \in L_2^m(0,1))$. Then for $n \to \infty$ functions $c^n(\vartheta, x)$, $P^n(\vartheta, x, y)$ and $h_n^2(\vartheta)$ uniformly converge, respectively, to functions $c(\vartheta, x)$, $P(\vartheta, x, y)$ and $h^2(\vartheta)$ for $\vartheta \in [\delta, T]$ and $x, y \in [0,1]$, where δ and T are arbitrary positive numbers.

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